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E_8 quiver gauge theory and mirror symmetry

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Abstract: We show that the Higgs branch of a four-dimensional Yang-Mills theory, with gauge and matter content summarised by an E_8 quiver diagram, is identical to the generalised Coulomb branch of a four-dimensional superconformal strongly coupled gauge theory with E_8 global symmetry. This is the final step in showing that there is a Higgs-Coulomb identity of this kind for each of the cases $\{0\}$, A_1, A_2, D_4, E_6, E_7 and E_8. This series of equivalences suggests the existence of a mirror symmetry between the quiver theories and the strongly coupled theories. We also discuss how to interpret the parameters of the quiver gauge theory in terms of the Hanany-Witten picture.

Keywords: 1/N Expansion, Duality in Gauge Field Theories, D-branes, String Duality.

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Contents

1. Introduction

The set of four-dimensional $N = 2$ superconformal SU(2)[1] Yang-Mills gauge theories with coupling $\tau = i/g^2 + \theta$ follows Kodaira's classification [1] of toroidal singularities (τ being the torus modulus). (We shall refer to these theories as Seiberg-Witten (SW) theories). This classification falls into an ADE pattern, and the type of singularity gives the global symmetry of the corresponding theory. There are A_n and D_n singularities at Im $\tau = \infty$, as well as a D_4 singularity that can occur at all values of τ. Furthermore, there are six singularities at finite values of τ, of types $\{0\}$, A_1, A_2, E_6, E_7 and E_8. Hence there are precisely seven strongly coupled superconformal theories, with global symmetries equal to the corresponding singularity type (see also [2, 3]). Of these theories the latter three have no lagrangian description, which makes them difficult to study.

However, a possible way to sidestep this difficulty appeared when [4] found the remarkable fact that the generalised Coulomb branches of the $\{0\}$, A_1, A_2, D_4 and E_6 theories are identical to the Higgs branches of other, a priori completely different gauge theories. These gauge theories, which we will call *quiver* theories, are four-dimensional $N = 2$ supersymmetric Yang-Mills theories with gauge and matter

[1] Here "SU(2)" refers to those of the theories that posess a lagrangian.

content that may be described by an ADE Dynkin diagram (or, rather, a *quiver* diagram). This Dynkin diagram is precisely that of the ADE global symmetry of the corresponding SW theory.

The authors of [5] extended the analysis to the E_7 case and, encouraged by these results, proposed the existence of a mirror symmetry between four-dimensional SW theories and quiver gauge theories, analogous to the mirror symmetry acting on three-dimensional gauge theories [6]. The mirror symmetry would exchange the generalised Coulomb branch of one theory with the Higgs branch of the other, and would provide a map between mass parameters of one theory and Fayet-Iliopoulos parameters of the other. Since the Coulomb branch receives quantum corrections but the Higgs branch does not, one consequence of the mirror symmetry is that quantum effects in one theory arise classically in the dual theory, and vice versa.

In this paper, we show the remaining moduli space identity, i.e. for the E_8 case, thus exhausting the set of SW theories. We also derive, at least implictly, the map between mass and Fayet-Iliopoulos parameters for the E_8 case.

The quiver gauge theory can be realised as the worldvolume theory of a D3-brane probing an orbifold singularity, and in section 2 we derive the gauge and matter content from such a setup, showing how to describe it by means of a quiver diagram. In section 3 we construct the curve describing the Higgs branch, expressed in gauge group invariants and Fayet-Iliopoulos parameters. In section 4 we attempt a deeper understanding of the quiver theory in terms of the Hanany-Witten picture [7], and finally, section 5 contains a summary and discussion.

2. The quiver Higgs branch

To construct the quiver gauge theory we start from type IIB string theory in ten flat dimensions (labelled $0, 1, \ldots, 9$) and make an orbifold $\mathbb{C}^2/\Gamma$ out of the 6789 directions [8]. If Γ is a discrete subgroup of SU(2), then (the non-compact) $\mathbb{C}^2/\Gamma$, with its single fixed point at the origin, may be viewed as a local description of a K3 orbifold near one of its several fixed points.

Then Γ may be chosen from one of the following groups: the cyclic groups $\mathbb{Z}_n$, the dihedral groups $\mathcal{D}_n$, the trihedral group $\mathcal{T}$, the octahedral group $\mathcal{O}$, and the icosahedral group $\mathcal{I}$. Depending on the choice of Γ, the resulting quiver theory will be associated with a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8, respectively. As we are interested in the last case here, we take $\Gamma = \mathcal{I}$.

We now probe the singularity at the origin of $\mathbb{C}^2/\Gamma$ by putting on it a D3-brane and its $|\Gamma| - 1$ images ($|\Gamma|$ = the order of Γ)[2] living in the 0123 dimensions. Open strings stretching between the branes provide a massless Neveu-Schwarz sector that

[2]One needs $|\Gamma|$ images to make a full representation of Γ.

consists of the states

$$A_\mu \equiv \lambda_V\, \psi^\mu_{-1/2}|0\rangle_{NS},\ \ \mu = 0,1,2,3$$

$$x^i\ \equiv \lambda_I\, \psi^i_{-1/2}|0\rangle_{NS},\ \ i = 4,5$$

$$x^m \equiv \lambda_{II}\, \psi^m_{-1/2}|0\rangle_{NS},\ m = 6,7,8,9$$

where $\psi^\mu_{-1/2}$ are the lowest NS raising modes appearing in the Laurent expansion of the string worldsheet fermions, and the (tachyonic) NS ground state $|0\rangle_{NS}$ has odd fermion number, $(-1)^F = -1$. The $|\Gamma| \times |\Gamma|$ Hermitian matrices λ_V, λ_I and λ_{II} are Chan-Paton matrices, required to obey invariance under Γ,

$$\gamma_\Gamma \lambda_V \gamma_\Gamma^{-1} = \lambda_V\,, \tag{2.1}$$

$$\gamma_\Gamma \lambda_I \gamma_\Gamma^{-1} = \lambda_I\,, \tag{2.2}$$

$$\begin{pmatrix} \gamma_\Gamma \lambda^1_{II} \gamma_\Gamma^{-1} \\ \gamma_\Gamma \lambda^2_{II} \gamma_\Gamma^{-1} \end{pmatrix} = G_\Gamma \begin{pmatrix} \lambda^1_{II} \\ \lambda^2_{II} \end{pmatrix}. \tag{2.3}$$

Here the matrices γ_Γ make up the regular representation of the action of Γ on the Chan-Paton indices, and G_Γ is some matrix in the 2×2 representation of Γ, acting on the two-vector $(\lambda^1_{II}, \lambda^2_{II})$.

The invariance conditions (2.1)–(2.3) break the original $\mathrm{U}(|\Gamma|)$ gauge group to a product of unitary subgroups, $F \equiv \prod_i \mathrm{U}(N_i)$,[3] and imply that the gauge field on the brane, A_μ, transforms in the adjoint of F. Moreover, the x^i make up a hypermultiplet in the adjoint of F, and the x^m make up a hypermultiplet transforming in the fundamentals of subgroups $\mathrm{U}(N_i) \times \mathrm{U}(N_j)$, as $(\mathbf{N}_i, \overline{\mathbf{N}}_j)$.[4] The x^i may be interpreted as the position of a brane in the 45 directions, and the x^m similarly parameterise motions of the brane in the 6789 directions.

For $\Gamma = \mathcal{I}$, the gauge group is $F = \mathrm{U}(1) \times \mathrm{U}(2)^2 \times \mathrm{U}(3)^2 \times \mathrm{U}(4)^2 \times \mathrm{U}(5) \times \mathrm{U}(6)$, and the x^m hypermultiplet[5] transforms as $(\mathbf{1}, \overline{\mathbf{2}}) \oplus (\mathbf{2}, \overline{\mathbf{3}}) \oplus (\mathbf{3}, \overline{\mathbf{4}}) \oplus (\mathbf{4}, \overline{\mathbf{5}}) \oplus (\mathbf{5}, \overline{\mathbf{6}}) \oplus (\mathbf{2}, \overline{\mathbf{4}}) \oplus (\mathbf{4}, \overline{\mathbf{6}}) \oplus (\mathbf{3}, \overline{\mathbf{6}})$. Comparing this information with an E_8 Dynkin diagram, we see that the matter and gauge content can be summarised by an extended Dynkin diagram, as in figure 1 [9]. We associate each subgroup $\mathrm{U}(N_i)$ with a node, letting the edges between them represent the x^m hypermultiplets. In addition, the edges are equipped with arrows to indicate the way in which the hypermultiplets transform, the direction being from the fundamental towards the antifundamental representation.[6]

[3]There is a diagonal U(1) which acts trivially on vectors and hypermultiplets, implying that the actual nontrivial gauge group is $F/\,\mathrm{U}(1)$.

[4]For this reason the x^m hypermultiplets are sometimes referred to as bifundamentals.

[5]More correctly, in $N = 1$ language the hypermultiplet is given by a pair $(\Phi, \overline{\Phi})$ of a chiral and an antichiral superfield. The chiral field transforms as described in the text and the antichiral transforms in the conjugate representation.

[6]Notice that a change of direction of an arrow only affects the final result by a change of sign in the corresponding FI parameter.

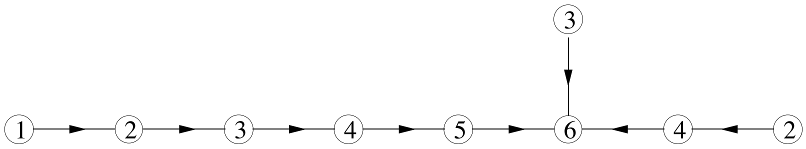

Figure 1: The E_8 extended Dynkin diagram.

The result is a *quiver* diagram [9], which, besides being a convenient gauge theory summary in general, will be useful to us when computing the curve that describes the Higgs branch (in section 3).

Due to the orbifolding of the 6789 directions one needs to take into account twisted states. We denote by a 3-vector $\vec{\phi}_k$ the triplet of NS-NS twisted sector scalar fields associated with the k:th U(1) generator of F, i.e. $k = 1, \ldots, 9$ labels the nodes of the quiver diagram. The low energy effective action on a D3-brane then includes the potential term

$$V \equiv \sum_{i,j} \mathrm{Tr}([x^i, x^j]^2) + \sum_{i,m} \mathrm{Tr}([x^i, x^m]^2) + \sum_{m,n} \mathrm{Tr}([x^m, x^n]^2) + H(x^m, x^n, \vec{\phi}_k)\,, \quad (2.4)$$

where H is a function of $\vec{\phi}_k$ and the x^m hypermultiplets, involving a term $(\vec{\phi}_k)^2$ and products of the form $\phi_k^\alpha x^m x^n$ (α labels the three components of $\vec{\phi}_k$). The last two terms of (2.4) may be rewritten using the definition [8]

$$\vec{\mu}_k \equiv \mathrm{Tr}\left[\lambda_V^k \left\{\varphi^{1\dagger}\vec{\sigma}\varphi^1 + \varphi^{2\dagger}\vec{\sigma}\varphi^2\right\}\right], \quad (2.5)$$

where $\varphi^1 = (z^1, -\bar{z}^{\bar{2}})$ and $\varphi^2 = (z^2, \bar{z}^{\bar{1}})$, with $z^1 \equiv x^6 + ix^7$ and $z^2 \equiv x^8 + ix^9$. The λ_V^k are generators of the k:th U(1) subgroup of F, and $\vec{\sigma} \equiv (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. Labelling the rest of the generators of F by the index a, we obtain

$$\sum_k (\vec{\mu}_k - \vec{\phi}_k)^2 + \sum_a (\vec{\mu}_a)^2 \quad (2.6)$$

for the last two terms of eq. (2.4).

The vacuum condition $V = 0$ yields the vacuum moduli space as two branches, the Coulomb branch and the Higgs branch. On the Coulomb branch we have $\vec{\phi}_k = 0$, $x^m = 0$ and $x^i \neq 0$, i.e. the D3-brane is stuck at the orbifold singularity in the 6789 directions but is free to move in the 45 directions. The Higgs branch, on the other hand, requires $x^i = 0$, implying that the D3-brane is stationary in the 45 directions, while the x^m may be nonzero. The kind of space that the x^m describe depends on the value of $\vec{\phi}_k$. If $\vec{\phi}_k = 0$ the Higgs branch is just the orbifold $\mathbb{C}^2/\Gamma$, whereas for non-vanishing $\vec{\phi}_k$ this hypermultiplet space is a resolved version of $\mathbb{C}^2/\Gamma$, with $\vec{\phi}_k$ controlling the size of the singularity blow-up. The latter space is an Asymptotically Locally Euclidean (ALE) space. That the Higgs branch takes this form may be seen by noting that the functions (2.5) are precisely the moment maps arising in the mathematical construction of hyper-Kähler quotients [10, 11].

It turns out that the twisted sector moduli $\vec{\phi}_k$ discussed above are just the Fayet-Iliopoulos (FI) parameters of the quiver theory. The classical vacua in gauge theories are determined by integrating out the auxiliary fields and requiring that the resulting potential vanish. In our case one obtains, for each U(1) generator of the gauge theory, a real equation from the D-terms and a complex equation from the F-terms. These together constitute the three equations

$$\vec{\mu}_k - \vec{\zeta}_k = 0\,, \tag{2.7}$$

where $\vec{\mu}_k$ is defined as in eq. (2.5) and the 3-vector $\vec{\zeta}_k \equiv (\zeta_k^R, \zeta_k^C, \overline{\zeta_k^C})$ denotes the triplet of FI terms associated with the k:th U(1) generator (R stands for "real" and C for "complex"). The condition (2.7) is precisely the vacuum requirement that the first term in eq. (2.6) vanish, if we identify $\vec{\phi}_k$ with $\vec{\zeta}_k$, an identification which is corroborated in section 4.

Thus we see that the FI parameters are associated with resolution of the orbifold singularity, and that there is one FI term for every node of the quiver diagram. In fact, the FI parameters may be defined as the period of the hyper-Kähler triplet $\vec{\omega}$ of complex forms,

$$\vec{\zeta}_k \equiv \int_{\Omega_k} \vec{\omega}\,, \tag{2.8}$$

where Ω_k is the k:th of the eight[7] 2-cycles required to blow up the $\mathbb{C}^2/\mathcal{I}$ singularity.[8]

3. The $\mathbf{E}_8$ calculations

The algebraic variety describing the Higgs branch of the quiver theory is defined by an equation involving three polynomials in the bifundamentals, that are invariant under the gauge group F [8]. Our goal here is to show that this curve is identical to the Seiberg-Witten (SW) curve describing the generalised Coulomb branch of the SW theory with E_8 global symmetry. The generalised Coulomb branch was defined in [5] as the fibration of the Seiberg-Witten torus over the ordinary Coulomb branch. The SW curve was found in [13], but we will use the results of [14], as their form of the curve is more useful for our purposes.

To find the algebraic curve we use the graphical method described in [4], which is based on the quiver diagram. Traces over the bifundamentals are represented by loops in the diagram, and the F-flatness conditions (2.7) are imposed as graphical rules for manipulating these traces. The rules are summarised in figure 2. Note that,

[7]The fact that one U(1) acts trivially leads to a relation among the FI parameters, see section 3.

[8]The $\mathbb{C}^2/\Gamma$ singularities fall into an ADE classification according to the pattern of intersecting 2-cycles required to resolve them [12]. These 2-cycles behave exactly like simple roots of the corresponding Dynkin diagram, and the classification matches the quiver diagrams so that the 2-cycles may be identified with the FI parameters through eq. (2.8).

JHEP05(2001)021

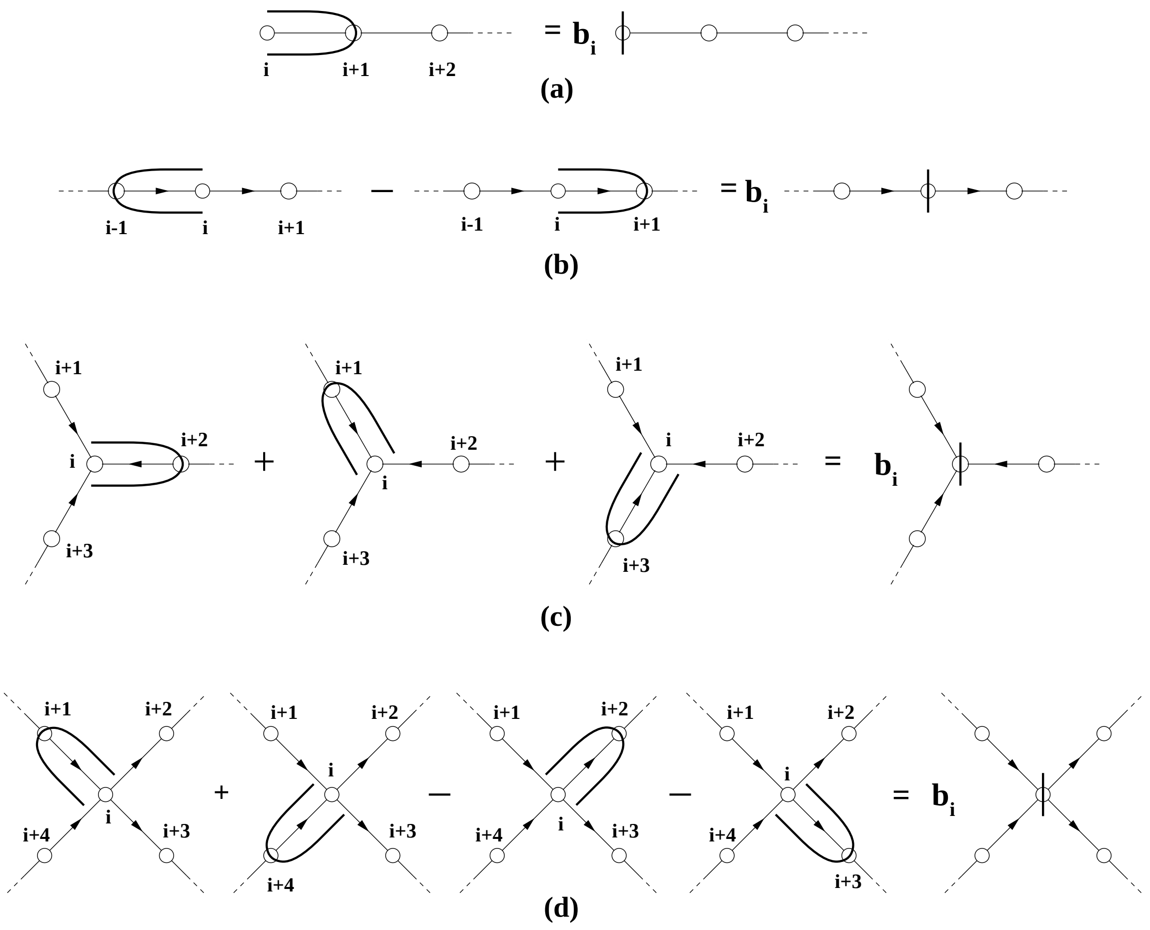

Figure 2: The rules to manipulate traces in a quiver diagram. The b_i are Fayet-Iliopoulos parameters.

by combining traces of these rules, we obtain a relation between the FI terms,

$$b_1 - 2b_2 - 3b_3 - 4b_4 - 5b_5 - 6b_6 - 4b_7 + 2b_8 + 3b_9 = 0\,, \tag{3.1}$$

which allows us to eliminate one of the b_i's.

To ultimately obtain a form of the curve that allows immediate comparison with the SW curve as given in [14], our approach is to express the square of the highest-order invariant in terms of lower-order invariants. For this we need the three F-invariants X, Y and Z, as well as some simpler matrices A, B and C that we define to simplify calculations; they are all given in figure 3.

Using the Schouten identity

$$\mathrm{Tr}(\{A,B\}C) = \mathrm{Tr}(AB)\,\mathrm{Tr}(C) + \mathrm{Tr}(AC)\,\mathrm{Tr}(B) + \mathrm{Tr}(BC)\,\mathrm{Tr}(A) - \mathrm{Tr}(A)\,\mathrm{Tr}(B)\,\mathrm{Tr}(C)$$

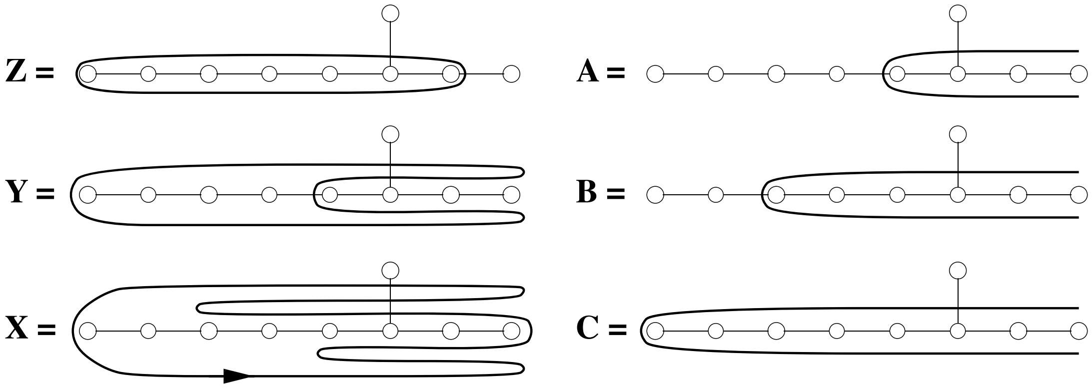

Figure 3: Invariants and matrices.

and the cyclic property of the trace we can reduce the square of the highest invariant, which, as can be seen from the diagram (using a U(1) Schouten identity), may be written as $X^2 = \mathrm{Tr}(ABCABC)$, to products of ($X$ and) traces of at most two of the matrices A, B, C,[9]

$$
\begin{aligned}
X^2 = {} & 2\,\mathrm{Tr}(ABC)\Big(\mathrm{Tr}(A)\,\mathrm{Tr}(BC) + \mathrm{Tr}(B)\,\mathrm{Tr}(AC) + \mathrm{Tr}(C)\,\mathrm{Tr}(AB) - \\
& \qquad - \mathrm{Tr}(A)\,\mathrm{Tr}(B)\,\mathrm{Tr}(C)\Big) + \\
& + \mathrm{Tr}(A^2)\left(\mathrm{Tr}(BC)^2 + \mathrm{Tr}(B)\,\mathrm{Tr}(C)\left[\frac{1}{2}\mathrm{Tr}(B)\,\mathrm{Tr}(C) - \mathrm{Tr}(BC)\right]\right) + \\
& + \mathrm{Tr}(B^2)\left(\mathrm{Tr}(AC)^2 + \mathrm{Tr}(A)\,\mathrm{Tr}(C)\left[\frac{1}{2}\mathrm{Tr}(A)\,\mathrm{Tr}(C) - \mathrm{Tr}(AC)\right]\right) + \\
& + \mathrm{Tr}(C^2)\left(\mathrm{Tr}(AB)^2 + \mathrm{Tr}(A)\,\mathrm{Tr}(B)\left[\frac{1}{2}\mathrm{Tr}(A)\,\mathrm{Tr}(B) - \mathrm{Tr}(AB)\right]\right) - \\
& - \frac{1}{2}\mathrm{Tr}(A^2)\,\mathrm{Tr}(B^2)\,\mathrm{Tr}(C^2) + \mathrm{Tr}(A)\,\mathrm{Tr}(B)\,\mathrm{Tr}(C)\times \\
& \times\left(\mathrm{Tr}(A)\,\mathrm{Tr}(BC) + \mathrm{Tr}(B)\,\mathrm{Tr}(AC) + \mathrm{Tr}(C)\,\mathrm{Tr}(AB)\right) - \\
& - \mathrm{Tr}(A)^2\,\mathrm{Tr}(BC)^2 - \mathrm{Tr}(B)^2\,\mathrm{Tr}(AC)^2 - \mathrm{Tr}(C)^2\,\mathrm{Tr}(AB)^2 - \\
& - \mathrm{Tr}(A)^2\,\mathrm{Tr}(B)^2\,\mathrm{Tr}(C)^2 - 2\,\mathrm{Tr}(AB)\,\mathrm{Tr}(AC)\,\mathrm{Tr}(BC)\,.
\end{aligned}
\tag{3.2}
$$

Having done this, we realize that in fact we need only to calculate a few traces. However, these calculations are nontrivial and the results are much too lengthy to fit in this paper,[10] so we only give the general form of the traces, as polynomials in

[9]Remark: For the Schouten identity to be useful for traces of an odd number of matrices, it is necessary that at least two of them are the same.

[10]All results can be downloaded from the web, in Maple V format, at
`http://www.physto.se/~brinne/E8quiver/`.

X, Y and Z,

$$\begin{aligned} \mathrm{Tr}(A) &= k_1 \\ \mathrm{Tr}(B) &= k_2 \\ \mathrm{Tr}(A^2) &= -2Z + k_3 \\ \mathrm{Tr}(C) &= k_4 Z + k_5 \\ \mathrm{Tr}(AB) &= k_6 Z + k_7 \\ \mathrm{Tr}(AC) &\equiv Y \\ \mathrm{Tr}(B^2) &= -2Y + k_8 Z + k_9 \\ \mathrm{Tr}(BC) &= Z^2 + k_{10} Y + k_{11} Z + k_{12} \\ \mathrm{Tr}(C^2) &= \mathrm{Tr}(C)^2 \\ \mathrm{Tr}(ABC) &= X \end{aligned} \tag{3.3}$$

The coefficients k_1, k_2, etc., are polynomials in the Fayet-Iliopoulos terms b_i, $i = 2, \ldots, 9$.[11]

We note here that the invariants X, Y and Z are of order 15, 10 and 6 in the b_i's, and the traces of A, B and C are of order 3, 5 and 7, respectively. From this information it is easy to find the orders of the coefficients defined in (3.3); for instance, the ones that are most difficult to compute,[12] k_9 and k_{12}, are polynomials of order 10 and 12, respectively, in eight variables.

To compare our curve (3.2) with the SW curve, we need to put it on its canonical form,

$$X^2 = -Y^3 + f(Z)Y + g(Z)\,, \tag{3.4}$$

where

$$\begin{aligned} f(Z) &= \omega_2 Z^3 + \omega_8 Z^2 + \omega_{14} Z + \omega_{20}\,, \\ g(Z) &= -Z^5 + \omega_{12} Z^3 + \omega_{18} Z^2 + \omega_{24} Z + \omega_{30}\,, \end{aligned}$$

and the ω_n are polynomials of order n in the b_i's. We accomplish this by plugging in the traces (3.3) into eq. (3.2) and shifting our variables X, Y, Z. The substitution of traces puts our curve on the form

$$\begin{aligned} X^2 - K_X X = {} & -Y^3 - Z^5 + (\alpha_{Y^2Z} Z + \alpha_{Y^2}) Y^2 + \\ & + (\alpha_{YZ^3} Z^3 + \alpha_{YZ^2} Z^2 + \alpha_{YZ} Z + \alpha_Y) Y + \\ & + \alpha_{Z^4} Z^4 + \alpha_{Z^3} Z^3 + \alpha_{Z^2} Z^2 + \alpha_Z Z + \alpha_0\,, \end{aligned}$$

where

$$K_X \equiv \beta_Y Y + \beta_{Z^2} Z^2 + \beta_Z Z + \beta_0$$

[11] b_1 was eliminated using eq. (3.1).

[12] When calculating k_9, we found it was necessary to use the Schouten identity to express $\mathrm{Tr}(A^3) = \frac{1}{2}\mathrm{Tr}(\{A,A\}A) = \frac{3}{2}\mathrm{Tr}(A)\,\mathrm{Tr}(A^2) - \frac{1}{2}\mathrm{Tr}(A)^2$.

JHEP05(2001)021

and the α's and β's are independent of the invariants X, Y and Z. To get rid of the X-term, we shift X by $-\frac{1}{2}K_X$. Expanding the result, we obtain a coefficient of the Y^2-term such that we must shift Y by $-\frac{1}{3}(\frac{1}{4}\beta_Y^2 + \alpha_{Y^2Z}Z + \alpha_{Y^2})$ in order to eliminate Y^2. Finally, the Z^4-term is substituted away by $Z \to Z - \frac{1}{5}(\frac{1}{4}\beta_{Z^2}^2 + \frac{1}{3}\alpha_{YZ^3}\alpha_{Y^2Z} + \alpha_{Z^4})$, and we end up with unwieldy expressions for the ω_n's, which should be compared to the Seiberg-Witten coefficients w_n of [14].

Finding the explicit relation between our FI parameters and the mass parameters w_n would be a next to impossible task unless we had a good idea of what it should be. Guided by our conjecture that the Higgs branch algebraic curve and the Seiberg-Witten curve are identical, and the fact that Noguchi et al. [14] used Casimir invariants to express their SW curve, we use E_8 Casimirs written in terms of the FI parameters. It is then straightforward, although very time consuming, to compare our coefficients with those of [14].

To find the Casimirs, first note that the simple roots of E_8 may, as we saw in section 2, be identified with the FI parameters via eq. (2.8),

$$b_i = \int_{\Omega_i} J\,,$$

where Ω_i is a 2-cycle behaving as a simple root, and J is the Kähler form on the orbifold. The Casimir invariants P_k may be found as [14]

$$P_k = 2^{-k} c_{240-k}\,,$$

where c_{240-k} is the coefficient of t^{248-k} in the characteristic polynomial

$$\det(t - v\cdot H) = \prod_{k=1}^{248}(t - v_k)\,, \tag{3.5}$$

where v_k are the weights of the fundamental representation, and $v \cdot H \equiv \mathrm{diag}(v_1, \ldots, v_{248})$. Factoring out t^{248} and defining

$$\chi_n \equiv \mathrm{Tr}\left[(v\cdot H)^n\right] = \sum_{k=1}^{248} v_k^n\,, \tag{3.6}$$

we may rewrite (3.5) as

$$\begin{aligned} t^{248}\prod_{k=1}^{248}(1 - \frac{v_k}{t}) &= t^{248}\exp\left\{\sum_{k=1}^{248}\ln\left(1 - \frac{v_k}{t}\right)\right\} \\ &= t^{248}\sum_{m=0}^{\infty}\left(-\frac{1}{m!}\right)^m\left(\sum_{n=1}^{\infty}\frac{\chi_n}{nt^n}\right)^m\,, \end{aligned} \tag{3.7}$$

a form which facilitates the extraction of coefficients. Writing the weights v_k in the usual way as linear combinations of simple roots (see appendix A), we thus obtain the Casimirs P_k expressed in FI parameters b_i.[13]

[13]The trace identities listed in appendix B were useful in this calculation.

To actually compare ω_n and w_n turned out to be quite demanding. The highest-order coefficient ω_{30} is a polynomial of order 30 in eight variables, which means that it has in principle 10295472 terms. We did our calculations in Maple V software on a Digital Alpha workstation, but since it would not store a polynomial of that size in its memory, much less compare two of them, we tried to find other software that would, without success. Below we describe the partly numerical check we performed, which, short of writing a dedicated program, seems the best one possible, as well as totally convincing.

We explicitly checked up to ω_{12} by comparing the full expressions for the polynomials and got complete agreement. The higher order ω's we could only check numerically and they agree when all the b_i's are set to random and different prime numbers. We have thus shown that the E_8 quiver Higgs branch is equal to the generalised Coulomb branch of the SW theory with E_8 global symmetry.

4. The Hanany-Witten picture

The IIB picture of D3-branes on a $\mathbb{C}^2/\mathbb{Z}_n$ orbifold singularity (which is of type A_{n-1}) is T-dual to a picture of type IIA string theory in a background of D4-branes stretching between NS5-branes [15]. This dual picture, the Hanany-Witten (HW) picture [7], provides an intuitive geometric interpretation of blow-ups of A_{n-1} type singularities. An analogous picture exists for D_n type singularities [16, 17], and it seems plausible that there are generalisations also to E_6, E_7 and E_8. In this section, we analyse the HW picture for the $\mathbb{C}^2/\mathbb{Z}_n$ case along the lines of [15] (see also [18, 19]); in particular we clarify the role of the Fayet-Iliopoulos terms.

Starting from the type IIB string theory configuration ($\times$ means the object is extended in that direction, and $-$ means it is pointlike)

	0	1	2	3	4	5	6	7	8	9
sing	×	×	×	×	×	×	−	−	−	−
D3	×	×	×	×	−	−	−	−	−	−

we T-dualise along the 6-direction to get

	0	1	2	3	4	5	6	7	8	9
NS5	×	×	×	×	×	×	−	−	−	−
D4	×	×	×	×	−	−	×	−	−	−

in type IIA string theory in flat spacetime. There are n NS5-branes, which all coincide in the 789 directions, but not necessarily in the 6-direction. Between them D4-branes are suspended, which are the T-duals of the IIB D3-branes. The rotational symmetry $\mathrm{SO}(3) \simeq \mathrm{SU}(2)$ of the 789 coordinates translates into the $\mathrm{SU}(2)_R$ symmetry of the gauge theory living on the D4-branes. The hypermultiplets arise from fundamental strings stretching across the NS5-branes, between neighbouring D4-branes.

JHEP05(2001)021

Resolving singularities in the IIB picture corresponds to separating NS5-branes along the 789 directions in the IIA picture. By an SU(2) rotation we can always pick the direction of displacement to be x^7. Note that such a displacement breaks the 789 rotational symmetry; that is, blowing up a singularity breaks the $\mathrm{SU}(2)_R$ symmetry. If we move some of the NS5-branes in this way, with the D4-branes still stuck to them, and then T-dualise along x^6 again, we do not regain the D3-brane picture. Rather, the now tilted D4-branes dualise to a set of D5-branes (with nonzero B-field) with their 67 worldvolume coordinates wrapped on 2-cycles. Shrinking these 2-cycles to zero size, each of the wrapped D5-branes is a fractional D3-brane, which cannot move away from the singularity. Thus a fractional D3-brane corresponds to a D4-brane whose ends are stuck on NS5-branes.

To move a fractional D3-brane, or, equivalently, a wrapped D5-brane, along the 6789 directions, we need to add $n-1$ images (under $\mathbb{Z}_n$), all associated with a 2-cycle each. The sum of the full set of 2-cycles is homologically trivial and can be shrunk to zero size. Then the collection of wrapped D5-branes will look like a single D3-brane that can move around freely in the orbifold. This procedure corresponds in the HW picture to starting out with a single D4-brane stretching between two of the n NS5-branes, and wanting to move the D4-brane (in the 7-direction, say) away from the NS5-branes, detaching its ends. In order not to violate the boundary conditions of the D4-brane, we then need to put one D4-brane between each unconnected pair of NS5-branes and join them at the ends. We then get a total of n D4-branes forming a single brane winding once around the periodic 6-direction. The D4-brane may now be lifted off the NS5-branes and move freely, corresponding to the free D3-brane in the T-dual picture.

We may also gain some insight concerning the role played by the FI parameters in the HW picture, from the worldvolume theory of a wrapped D5-brane on the orbifold singularity. Consider such a brane living in the 012367 directions, with its 67 worldvolume coordinates wrapped on a 2-cycle Ω_k. The Born-Infeld and Chern-Simons terms in the worldvolume action are, schematically, [9]

$$
\begin{aligned}
I_{D5} = \int d^6x\sqrt{\det(g+\mathcal{F})} + \mu\int C^{(6)} + \mu\int C^{(4)}\wedge\mathcal{F} + \mu\int C^{(2)}\wedge\mathcal{F}\wedge\mathcal{F} + \\
+\mu\int C^{(0)}\wedge\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}\,, \qquad (4.1)
\end{aligned}
$$

where g is the metric on the worldvolume, $C^{(p)}$ is the R-R p-form, μ is a constant, and $\mathcal{F} = F^{(2)} + B^{NS}$ where the 2-form $F^{(2)}$ is the field strength of the gauge field on the brane and B^{NS} is the NS-NS 2-form on the brane. Dimensional reduction to the 0123 directions, by integrating over the 2-cycle, puts the first term of (4.1) on the form

$$
\int_{\Omega_k} d^2x\sqrt{\det(g_2+\mathcal{F}_2)}\int d^4x\sqrt{\det(g_4+\mathcal{F}_4)}\,, \qquad (4.2)
$$

where $\mathcal{F}_2 = C^{(2)} + B^{NS}$, g_2 is the metric on the 67 directions, and g_4 is the metric on the 0123 directions. Expanding (4.2) we obtain the coupling constant g_k^{-2} in four dimensions as the coefficient of $\int d^4x F_{\mu\nu}F^{\mu\nu}$. It is just the factor on the left in (4.2), which we can write as

$$g_k^{-2} = \left| \int_{\Omega_k} \left(B^{NS} + iJ \right) \right| . \tag{4.3}$$

In the HW picture the coupling constant of the four dimensional theory is proportional to the length of the D4-brane in the additional fifth direction of the brane. Hence (4.3) measures the total distance between two NS5-branes between which the D4-brane is suspended. Furthermore, since the distance between the NS5-branes in the isometry direction (in our case x^6) is given by the flux of the B^{NS} field on the corresponding cycle, we have to interpret $\int_{\Omega_k} J$ as the position of the NS5-branes in a direction orthogonal to that, let us choose x^7. Movement of the NS5-branes in the remaining directions x^8 and x^9 now corresponds to turning on the $\mathrm{SU}(2)_R$ partners of the Kähler form.

The integral of J over a 2-cycle is also, by definition, a Fayet-Iliopoulos term. A hyper-Kähler manifold has an SU(2) manifold of possible complex structures. Choosing a complex structure we can define the Kähler form J as ω^1, and the holomorphic 2-form as $\omega^2 + i\omega^3$. These three 2-forms rotate into each other under $\mathrm{SU}(2)_R$ transformations, corresponding to choosing a different complex structure. The k:th triplet of FI terms is defined by the period of $\vec{\omega} = (\omega^1, \omega^2, \omega^3)$ (and hence also transforms as a triplet under $\mathrm{SU}(2)_R$), as

$$\vec{\zeta}_k \equiv \int_{\Omega_k} \vec{\omega} \, .$$

Hence

$$\zeta_k^R = \int_{\Omega_k} J \, ,$$

where ζ_k^R is the real component of the triplet of FI terms $\vec{\zeta}_k = (\zeta_k^R, \zeta_k^C, \overline{\zeta_k^C})$.

Another way to obtain the FI terms of the four-dimensional Yang-Mills theory is via dimensional reduction and supersymmetrisation of the D5-brane worldvolume theory [9]. The third term of (4.1) can be rewritten as

$$\int d^6x (A_\mu - \partial_\mu c^{(0)})^2 \, ,$$

where $c^{(0)}$ is the Hodge dual potential of $C^{(4)}$ in six dimensions. After integration over the k:th 2-cycle we supersymmetrise this to

$$\int d^4x d^4\theta (\mathbf{C}_k - \overline{\mathbf{C}}_k - \mathbf{V})^2 \, ,$$

where $\mathbf{C}_k$ is a chiral superfield whose complex scalar component is $c^{(0)} + i\zeta_k^R$, and $\mathbf{V}$ is the vector superfield containing A_μ. Here the imaginary part ζ_k^R of the scalar component is the real FI term in four dimensions, and we see that it arises as the superpartner of $c^{(0)}$.

5. Conclusions

In this paper we have discussed the conjectured mirror symmetry between the Higgs branch of quiver gauge theories and the generalised Coulomb branch of certain four-dimensional strongly coupled, globally symmetric gauge theories. This symmetry was hinted at in [4], where equivalence of the E_6 curves was established, and it was substantiated in [5], where equivalence of the E_7 curves was presented and the conjecture was made precise. Here we have shown the final equivalence, that of the E_8 curves for the two branches.

The proof we have given is by direct comparison. It would be nice to find a proof such as a chain of dualities leading from one model to the other, as indicated in [5]. There are obstacles to this in that, e.g., many of the dualities expected to enter that chain are not known.

In this context we emphasize an important aspect of our results, namely as a guide for finding Hanany-Witten type constructions. Assume that one is contemplating a HW picture of NS5-branes with D4-branes ending on them and that this is supposed to describe the IIA T-dual of D3-branes on an (E_6, E_7 or) E_8 singularity in IIB. Moving the NS5-branes in the HW picture corresponds to blowing up the singularity in the dual picture. As discussed in section 4, the FI parameters give the position of the NS5-branes in the HW picture when they are moved. Since we have given the relation of the FI parameters to the parameters governing the deformation of the algebraic variety, one may now check that the possible motions on the HW side (allowed by the particular geometry suggested) correspond to the known allowed deformations.

We have performed the above check for some of the known dualities and hope to use it in future efforts to find HW pictures of the E_n-theories.

Finally we mention that all is set up for finding the remaining trace identities in appendix B. The calculations will be carried out as soon as the computer capacity becomes available to us.

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A. The $\mathbf{E_8}$ weights

The E_8 Casimir invariants may be expressed in terms of fundamental weights as shown in section 3. The weights may in turn be expressed in terms of the E_8 simple roots.[14]

[14] Although we treat the case E_8 here, the procedure is quite general.

We want to find the 248 weights of the fundamental representation of E_8. This is the same as the adjoint representation, and it is real, whence half of the weights are just minus the first half. In addition one finds that eight of them are zero, so we end up with only 120 nonvanishing independent weights.

Viewing the simple roots b_i as 8-vectors, the Cartan matrix for E_8,

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix},$$

may be viewed as the matrix of scalar products $b_i \cdot b_j$. If we start with a highest weight vector w_0, lower weights are obtained by subtracting simple roots whose scalar products with w_0 are positive [23]. To be explicit, choose w_0 such that its scalar product with b_1 is 1 and otherwise zero, i.e. w_0 would be represented by a row $[1, 0, 0, 0, 0, 0, 0, 0]$ in the Cartan matrix. Then the only possible next weight is $w_0 - b_1$, represented by $[-1, 1, 0, 0, 0, 0, 0, 0]$ in the Cartan matrix. Iterating this procedure, subtracting simple roots whose scalar products with the previous weight are positive, we obtain 120 nonvanishing weights w_k in "Cartan representation". To express these in terms of simple roots, we invert the Cartan matrix and compute v_k as

$$v_k = 2 \sum_{j=1}^{8} \sum_{i=1}^{8} w_k^j C_{ji} b_i \,,$$

where w_k^j denotes the j:th component of the vector w_k, C is the inverse of the Cartan matrix, and the factor 2 is a normalisation.

Note the slight change in convention; all the b_i's here have the same sign, whereas two of our FI terms in section 3 (b_8 and b_9) have opposite sign relative to the others.

B. Trace identities

The traces $\chi_n \equiv \mathrm{Tr}[(v \cdot H)^n]$ (eq. (3.6)) used in writing the Casimir invariants in section 3 satisfy identities that simplify the calculations slightly. Such trace identities were derived in [24] for the simple Lie algebras up to E_6, whereas the E_7 identities were calculated by the authors of [5]. For E_8 we found the following identities,

$$\chi_4 = \frac{1}{100}\chi_2^2$$
$$\chi_6 = \frac{1}{7200}\chi_2^3$$

$$\chi_{10} = \frac{1}{16}\chi_2\chi_8 - \frac{1}{69120000}\chi_2^5$$

Although straightforward in principle, due to limitations of Maple, we were not able to derive the rest of the trace identities, i.e. those expressing χ_{16}, χ_{22}, χ_{26} and χ_{28} in terms of the independent χ_n's.

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